Symplectic Mappings

John D. Hadjidemetriou
Department of Physics, University of Thessaloniki, Greece

Abstract. A widely used method to study the evolution of a Hamiltonian system of n degrees of freedom in phase space is by the method of Poincaré mapping on a surface of section. The continuous flow in the $2n$-dimensional phase space is reduced to a discreet flow in a $(2n-2)$-dimensional phase space, without loosing degrees of freedom. Evidently, the Poincaré mapping of a Hamiltonian system is symplectic. The difficulty is that it involves the solution of the system, usually by numerical integrations.

Another method used widely is the method of averaging, which is especially useful for the study of resonant motion. By this method, the "fast" angles are eliminated and this results to a decrease of the dimensions of the phase space, as with the Poincaré mapping. The flow is also symplectic, but information is lost due to the loss of one or more degrees of freedom.

The above methods are closely connected. Two methods have been developed recently for the construction of mapping models, for nearly integrable Hamiltonian systems of the form $H_0 + \epsilon H_1$, where H_0 in the integrable part. Both are based on an averaged Hamiltonian and though they are different, their basic philosophy is the same: the lost degrees of freedom in the averaged model are regained, thus recovering the information. In the first method (Wisdom, 1983,85) the high frequency terms eliminated in the averaging process are reintroduced, through periodic delta functions, thus recovering the original degrees of freedom. The solution obtained, considered at integral multiples of a basic period, define the mapping. In the second method (Hadjidemetriou 1991,93) the Poincaré mapping of the unperturbed Hamiltonian H_0 is obtained analytically and is expressed through a generating function which is given explicitly. Then the mapping is perturbed, by perturbing the generating function, in such a way that the new mapping has all the essential topological features of the original system. The perturbation is based on the averaged Hamiltonian and the mapping is exactly symplectic by its construction. In addition to these general methods, some other symplectic maps are also presented, for particular cases (e.g. cometary motion).

A basic problem related to the above mapping models is how good is the averaged model. A simple test is presented to check the range of validity of the averaged model and a method to correct it and introduce any missing features (i.e. missing high eccentricity resonances) is explained, even if the perturbation series do not converge. The effects of the use of a wrong or incomplete averaged model on the evolution of the system are discussed. Since the properties of all similar dynamical systems are essentially the same, as far as the evolution is concerned, the different mapping models will have the same behaviour and also the same properties as the original system, provided that the topology of their phase space is the same. The important point is to conserve the topology of the phase space of the original system in constucting the mapping model.

We shall also show how a symplectic map is used to study the long term evolution of a Hamiltonian system and give insight on the factors that govern the evolution and on the mechanism of generation of chaos through the interaction between the different degrees of freedom.